Conclusion

There are three clear pictures of ship waves in Ref. 4. The ships in the pictures are blunt and seem to be running at high speed. In the second and third pictures, no transverse wave is seen. In the first picture, however, the transverse waves are seen downstream. It appears that the transverse waves come from the stern, not from the bow.

Because of the shadow region, even in the case of a slender ship the transverse waves are in the region between the envelope and the boundary ray, whereas the divergent waves are between the ship boundary and the envelope. Because the transverse waves diffract into the shadow region, the shadow region is not free of the transverse waves. Far downstream the flow becomes uniform. Hence all of the rays become straight and all waves are those of the Kelvin wave. The slope of the envelope far downstream may be the same as that of the half wedge angle, 19.47 deg of the Kelvin wave.

When the ship form is blunt, all of the waves from the bow are divergent. In that case the slope of the boundary ray far downstream is less than that of the 19.47 deg Kelvin wave. It should be noted that the present theory applies only at low speeds.

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Force and Moment in Incompressible Flows

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Introduction

THE central problem of external fluid dynamics is the prediction and/or measurement of the force and moment exerted upon a body immersed in an arbitrary flowfield by the surrounding fluid. Although complete knowledge (either analytical or numerical) of the flowfield allows evaluation of this force and moment by integrating, over the surface of the body, the elemental contributions due to pressure and viscous shear, it is certainly of great theoretical interest and, possibly, of practical usefulness to obtain general alternate expressions for such quantities. This is particularly true for the case of incompressible flows, of interest here, for which the elimination of pressure from the formulas expressing the force and moment is desirable, due to the peculiar character of this variable. At present, particular formulas are available, e.g., for the force acting on a sphere immersed in a viscous steady creeping flow (see Ref. 1, p. 233) and for the force acting on an arbitrary body immersed in a potential unsteady flow (see Ref. 2, p. 300). Moreover, for the case of inviscid flows past an arbitrary two-dimensional body, Blasius has derived general relationships for both the aerodynamic force and moment (see, e.g., Ref. 1, p. 433). However, more general expressions are not available to date and it seems worthwhile to attempt to derive formulas that are valid for any flow regime (i.e., for arbitrary values of the Reynolds number) and, therefore, are capable of reobtaining all previously mentioned results within a general framework and from a unified perspective, as done in the present Note.

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Pressure Boundary Values Relationship

Let us consider an incompressible viscous flow which is governed by the time-dependent Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\nabla p-\nu\nabla\times\nabla\times\boldsymbol{u} \tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0 \tag{2}$$

where $\boldsymbol{u}$ is the velocity vector and p the pressure (per unit density) of the fluid. Equations (1) and (2) are supplemented by the initial and boundary conditions

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0 \tag{3}$$

$$\boldsymbol{u}|_S=\boldsymbol{b} \tag{4}$$

where S is the boundary of the region V occupied by the fluid. The initial velocity field $\boldsymbol{u}_0$ is assumed to be solenoidal, i.e.,

$$\nabla\cdot\boldsymbol{u}_0=0 \tag{5}$$

and the velocity $\boldsymbol{b}(\boldsymbol{r},t)$, prescribed at any point $\boldsymbol{r}\in S$, is assumed to satisfy the condition

$$\int \mathrm{d}S\boldsymbol{n}\cdot\boldsymbol{b}=0 \tag{6}$$

which follows from Eq. (2) by the application of Green's formula, $\boldsymbol{n}$ denoting the outward normal unit vector. In what follows it will be assumed that the problem defined by Eqs. (1-6) has a solution $(\boldsymbol{u},p)$ and that the vector fields $\boldsymbol{u}$ and ∇p belong to L^2; L^2 denoting the Hilbert space of square summable vector functions, defined in V, i.e., fields $\boldsymbol{v}$ for which the integral $\int \mathrm{d}V|\boldsymbol{v}|^2$ is finite. Basic to the derivation of a relationship involving pressure boundary values is a decomposition theorem of the Hilbert space L^2 into three orthogonal subspaces (see Ref. 3, pp. 15-17) which allows elimination of the pressure at the boundary in favor of the velocity field. This is achieved by first projecting the momentum equation (1) onto one of the subspaces $(H_1\equiv\{\boldsymbol{v}\in L^2\,|\boldsymbol{v}=\nabla\eta,\ \nabla^2\eta=0\}$, i.e., by taking the Hibert scalar product of Eq. (1) by any $\nabla\eta\in H_1$, to obtain,

$$-\int \mathrm{d}V\nabla p\cdot\nabla\eta=\int \mathrm{d}V\left\{\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nu\nabla\times\nabla\times\boldsymbol{u}\right\}\cdot\nabla\eta \tag{7}$$

Then, by transforming the volume integral on the left-hand side of Eq. (7) into a surface integral by means of the equation $\nabla p\cdot\nabla\eta=\nabla\cdot(p\nabla\eta)$, and rewriting the linear terms in the right-hand side of Eq. (7) as surface integrals by virtue of the identities $\boldsymbol{u}\cdot\nabla\eta=\nabla\cdot(\boldsymbol{u}\eta)$ and $(\nabla\times\nabla\times\boldsymbol{u})\cdot\nabla\eta=\nabla\times\boldsymbol{\zeta}\cdot\nabla\eta=\nabla\cdot(\boldsymbol{\zeta}\times\nabla\eta)$, where $\boldsymbol{\zeta}=\nabla\times\boldsymbol{u}$ is the vorticity, the following final equation is obtained easily:

$$-\int dSpn\cdot\nabla\eta=\int dV\{(u\cdot\nabla)u\}\cdot\nabla\eta$$

$$+\nu\int dSn\times\zeta\cdot\nabla\eta+\int dSn\cdot\frac{\partial b}{\partial t}\eta \tag{8}$$

Force and Moment on a Fixed Rigid Body

Let us now consider a rigid body immersed in an arbitrary incompressible flow. Equation (8) enables us to obtain closed formulas for the force and the moment exerted on the body in terms of the global solenoidal velocity field, without any reference to the pressure variable. Let S_1 be the surface of the body and S_2 an arbitrary surface enclosing the body and delimiting the considered domain of the flowfield. In a coordinate system fixed with respect to the body, we have

$$u|_{S_1}=b_1 \tag{9a}$$

$$u|_{S_2}=b_2 \tag{9b}$$

The total force f (per unit density of the fluid) exerted on the body is given by

$$f=\int_{S_1}dS(np+\nu n\times\zeta) \tag{10}$$

(see, e.g., Ref. 1, p. 178). Let us consider the component f_x of the force in the direction of the unit vector $\hat{x}$. The quantity $\hat{x}\cdot\int dS_1np\equiv\int dS_1\hat{x}\cdot np$ can be calculated from Eq. (8) by introducing the harmonic function η_x which satisfies the following Neumann boundary conditions:

$$n\cdot\nabla\eta_x|_{S_1}=-n\cdot\hat{x} \tag{11a}$$

$$n\cdot\nabla\eta_x|_{S_2}=0 \tag{11b}$$

to give

$$f_x=\int dV\{(u\cdot\nabla)u\}\cdot\nabla\eta_x+\nu\int_{S_1+S_2}dSn\times\zeta\cdot\nabla\eta_x$$

$$+\int_{S_1+S_2}dSn\cdot\frac{\partial b}{\partial t}\eta_x+\nu\hat{x}\cdot\int_{S_1}dSn\times\zeta \tag{12}$$

Since the velocity vanishes at the body surface, the second integral on S_1 is zero, so that, by regrouping the two remaining integrals on S_2 into a single term, Eq. (12) assumes the following final form:

$$f_x=\int dV\{(u\cdot\nabla)u\}\cdot\nabla\eta_x+\nu\left\{\int_{S_1}dSn\times\zeta\cdot(\nabla\eta_x+\hat{x})\right.$$

$$\left.+\int_{S_2}dSn\times\zeta\cdot\nabla\eta_x\right\}+\frac{d}{dt}\int_{S_2}dSn\cdot b_2\eta_x \tag{13}$$

Let us now consider the problem of evaluating the moment (per unit density) acting on the body, namely,

$$\tau=\int_{S_1}dSr\times(np+\nu n\times\zeta) \tag{14}$$

In this case the evaluation of the component of τ along a given direction, say $\hat{z}$, requires the introduction of a new harmonic function η_z satisfying the boundary conditions

$$n\cdot\nabla\eta_z|_{S_1}=-n\cdot z\times r \tag{15a}$$

$$n\cdot\nabla\eta_z|_{S_2}=0 \tag{15b}$$

By means of η_z, the same procedure as before easily can be shown to provide

$$\tau_z=\int dV\{(u\cdot\nabla)u\}\cdot\nabla\eta_z+\nu\left\{\int_{S_1}dSn\times\zeta\cdot(\nabla\eta_z+\hat{z}\times r)\right.$$

$$\left.+\int_{S_2}dSn\times\zeta\cdot\nabla\eta_z\right\}+\frac{d}{dt}\int_{S_2}dSn\cdot b_2\eta_z \tag{16}$$

It is noteworthy that the expressions of the force and the moment are quite similar, the only difference being in the harmonic function to be used for their evaluation.

Applications to Special Flow Regimes

Equations (13) and (16) are valid for steady as well as unsteady flows at arbitrary values of the Reynolds number. In particular, they contain the limit cases of special flow regimes. As a matter of example, the steady creeping flow and the unsteady inviscid flow will be considered, to reobtain the classical expressions of the drag exerted on a sphere by a uniform far field flow.

Let us consider a laminar flow having a uniform velocity $U=\hat{x}U$ at a large distance from a stationary sphere of radius a. If the flow is assumed to be symmetric with respect to the axis parallel to U and passing through the center of the sphere, the force acting upon the sphere is parallel to U and it is convenient to choose a spherical coordinate system (r,θ,ϕ) with the origin placed at the center of the sphere such that the direction $\theta=0$ coincides with the direction of U. In order to calculate f_x from Eq. (13), where S_1 is the surface of the sphere, we choose S_2 to be the concentric sphere of infinite radius, so that the harmonic function η_x satisfying the boundary conditions Eq. (11) easily is found to be

$$\eta_x=\tfrac{1}{2}a^3\cos\theta/r^2 \tag{17}$$

Consider now the case of a steady creeping flow. Equation (13) simplifies to

$$f_x=\nu\left\{\int_{S_1}dSn\times\zeta\cdot(\nabla\eta_x+\hat{x})+\int_{S_2}dSn\times\zeta\cdot\nabla\eta_x\right\} \tag{18}$$

since the volume integral can be neglected because $\nu\to\infty$ and the last surface integral vanishes, due to the assumption $dU/dt=0$. On the other hand, the flowfield satisfying the boundary conditions on the sphere and at infinity can be calculated analytically; the vorticity field is found to be (see, e. g., Ref. 1, p. 230)

$$\zeta=-\frac{3}{2}a\frac{U\times r}{r^3}=\frac{3}{2}aU\frac{\sin\theta}{r^2}\hat{\phi} \tag{19}$$

where $\hat{\phi}$ is a unit vector. Due to Eqs. (17) and (19), the surface integral over S_2 vanishes, so that Eq. (18) becomes

$$f_x=\nu\int_{S_1}dSn\times\zeta\cdot(\nabla\eta_x+\hat{x})$$

$$=\nu\int_{S_1}dS(-\hat{r})\times\left(\frac{3}{2}aU\frac{\sin\theta}{r^2}\hat{\phi}\right)\cdot(\nabla\eta_x+\hat{x})$$

$$=\frac{3}{2}\nu aU\int_{S_1}dS\frac{\sin\theta}{r^2}\hat{\theta}\cdot(\nabla\eta_x+\hat{x})$$

where $\hat{r}$ and $\hat{\theta}$ are unit vectors. By means of Eq. (17) we have

$$\hat{\theta}\cdot(\nabla\eta_x+\hat{x})=-((a^3/2)\sin\theta/r^3+\sin\theta)$$

so that

$$f_x = -\frac{3}{2}\nu a U \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin^3\theta \frac{3}{2}$$

and, after evaluating the integrals,

$$f_x = -6\pi\nu a U \tag{20}$$

which is the classical Stokes' law for the sphere.

Coming now to the case of inviscid unsteady flow, i.e., $\nu \to 0$, Eq. (13) provides

$$f_x = \int \mathrm{d}V\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}\cdot\nabla\eta_x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_2}\mathrm{d}S\boldsymbol{n}\cdot\boldsymbol{b}_2\eta_x \tag{21}$$

For the present case of potential flow, $\boldsymbol{u} = \nabla\phi$, with ϕ solution of the Neumann problem

$$\nabla^2\phi = 0 \tag{22a}$$

$$\boldsymbol{n}\cdot\nabla\phi|_S = \boldsymbol{n}\cdot\boldsymbol{b} \tag{22b}$$

It is then possible to recast Eq. (21) in the form

$$f_x = -\hat{x}\cdot\left(\frac{1}{2}\int_{S_1}\mathrm{d}S\boldsymbol{n}|\nabla\phi|^2 + \int_{S_1}\mathrm{d}S\boldsymbol{n}\frac{\partial\phi}{\partial t}\right) \tag{23}$$

where the first term is obtained by writing the convection term in Lamb's form and the second term is derived by Green's theorem, taking full advantage of the harmonic character of η_x and ϕ and using the respective boundary conditions Eqs. (11) and (22). Equation (23) coincides with the classical formula for irrotational unsteady flow (see, e.g., Ref. 1, p. 404). If we wish to obtain the drag on the sphere in this case directly from Eq. (21), a simple integration leads to the following final result [the volume integral is easily seen to vanish due to the forward-backward symmetry of the velocity field defined by problem Eq. (22)]

$$f_x = -\frac{2}{3}\pi a^3\frac{\mathrm{d}U}{\mathrm{d}t} \tag{24}$$

where the factor multiplying $\mathrm{d}U/\mathrm{d}t$ is the well-known apparent mass (per unit density) of the sphere (see, e.g., Ref. 2, p. 291).

Conclusion

General formulas for the force and moment acting on a rigid body, immersed in an incompressible flow, have been obtained. Though somewhat difficult to use, insofar as they require the knowledge of the entire solenoidal velocity field, these expressions have a general theoretical interest per se and might be of practical use in applications. For most numerical solutions of flows at intermediate to high Reynolds numbers, the evaluation of the pressure at the body surface gives rise to highly inaccurate results. A more accurate evaluation of the force and moment on the body is possible, therefore, by means of the present formulas, which do not involve the pressure at the body surface, but the velocity field in its entirety.

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Optimum Sensitivity Derivatives of Objective Functions in Nonlinear Programming

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Introduction

OPTIMUM sensitivity analysis is a technique which permits investigation of the sensitivity of an optimization problem's solution to variations of the problem's parameters. It yields derivatives of the optimum values of the design variables and objective function with respect to the parameters. These derivatives may then be used to perform trade-off analyses. Henceforth, these derivatives will be called sensitivity derivatives to distinguish them from the derivatives of objective function and constraints with respect to the design variables that are termed behavior derivatives.

This technique was introduced recently as a tool for structural synthesis.[1] It has been specialized to a formulation of the problem of structural sizing for minimum weight based on approximation concepts and dual methods,[2] and has been used in the optimization of damage tolerant structures.[3] Finally, it is one of the building blocks for a proposed multilevel approach to the design of large engineering systems.[4]

As shown in Ref. 1, a complete sensitivity analysis requires the following quantities as input: 1) the first- and second-behavior derivatives, 2) the first derivatives of behavior and behavior derivatives with respect to the parameters, and 3) the Lagrange multipliers. These need to be evaluated at the optimum point. Among these quantities, the first-behavior derivatives and the Lagrange multipliers are either by-products of the optimization itself or may be evaluated relatively inexpensively; but, a relatively significant computational cost has to be incurred to calculate the second derivatives. With this as motivation, the present Note shows that the second derivatives may be eliminated completely from the input of the optimum sensitivity analysis, provided that one accepts its curtailment to the first-order sensitivity derivatives of the objective function. Also, it is shown that when a complete first-order sensitivity analysis is performed, second-order sensitivity derivatives of the objective function are available at little additional cost.

First-Order Sensitivity Derivative of the Objective Function

Assume that we start from an optimization problem defined by

$$\min_X F(X,P) \tag{1a}$$

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